ON THE INTERPOLATION CONSTANT FOR SUBADDITIVE OPERATORS IN ORLICZ SPACES

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Abstract. Let $1 \le p < q \le \infty$ and let T be a subadditive operator acting on L^p and L^q. We prove that T is bounded on the Orlicz space L^φ, where $\varphi^{-1}(u) = u^{1/p}\rho(u^{1/q-1/p})$ for some concave function ρ and

$$\|T\|_{L^\varphi \to L^\varphi} \le C \max\{\|T\|_{L^p \to L^p}, \|T\|_{L^q \to L^q}\}.$$

The interpolation constant C, in general, is less than 4 and, in many cases, we can give much better estimates for C. In particular, if $p = 1$ and $q = \infty$, then the classical Orlicz interpolation theorem holds for subadditive operators with the interpolation constant $C = 1$. These results generalize our results for linear operators obtained in [7].

1. Introduction

Let (Ω, Σ, μ) be a complete σ-finite measure space. A Banach lattice X on (Ω, Σ, μ) is a Banach space of (equivalence classes of μ-a.e. equal) real or complex-valued functions on Ω such that if $|x(t)| \le |y(t)|$ μ-a.e. where $y \in X$ and x is μ-measurable, then $x \in X$ and $\|x\|_X \le \|y\|_X$. Lebesgue and Orlicz spaces are examples of Banach lattices.

Let $\varphi : [0, \infty) \to [0, \infty]$ be an *Orlicz function*, that is, a nondecreasing convex function such that $\varphi(0) = 0$ and $\lim_{u \to 0+} \varphi(u) = 0$ but not identically zero or infinity on $(0, \infty)$. For a measurable real or complex-valued function x, define a functional (*modular*)

$$I_\varphi(x) := \int_\Omega \varphi(|x(t)|)\, d\mu(t) = \int_0^\infty \varphi(x^*(s))\, ds, \tag{1}$$

where x^* is the non-increasing rearrangement of the function x. The second equality in (1) means that the modular is rearrangement invariant. The *Orlicz space* $L^\varphi = L^\varphi(\Omega, \Sigma, \mu)$ is the set of all equivalence classes of μ-measurable functions on Ω such that $I_\varphi(\lambda x) < \infty$ for some $\lambda = \lambda(x) > 0$. This space is a Banach space with two norms: the *Luxemburg-Nakano norm*

$$\|x\|_\varphi := \inf\{\lambda > 0 \ : \ I_\varphi(x/\lambda) \le 1\},$$

and the *Orlicz norm* (in the Amemiya form)

$$\|x\|_\varphi^0 := \inf_{k>0} \frac{1}{k}[1 + I_\varphi(kx)].$$

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Since $\|x\|_\varphi = \inf_{k>0} \frac{1}{k} \max\{1, I_\varphi(kx)\}$ it follows that

$$\|x\|_\varphi \le \|x\|_\varphi^0 \le 2\|x\|_\varphi.$$

The Orlicz space L^φ equipped with each of the above two norms is a rearrangement-invariant space, sometimes also called a symmetric space with the Fatou property. For general properties of Orlicz spaces we refer to the books [2, 8, 11, 14].

It is well known that any two Banach lattices X_0 and X_1 on the same measure space (Ω, Σ, μ) forms a Banach couple (X_0, X_1) in the sense of interpolation theory (see [9, p. 42]). The intersection $X_0 \cap X_1$ and the sum $X_0 + X_1$ of these two spaces are also Banach lattices on (Ω, Σ, μ) with the standard norms (cf. [2, 4, 9]). An operator T mapping $X_0 + X_1$ into itself is said to be *subadditive* if for all $x, y \in X_0 + X_1$,

$$|T(x+y)(t)| \le |Tx(t)| + |Ty(t)| \quad \mu\text{-a.e. on } \Omega.$$

If, in addition, we have also that $|T(\lambda x)(t)| = |\lambda|\,|Tx(t)|$ μ-a.e. on Ω for any $x \in X_0 + X_1$ and any scalar λ, then the operator T is called *sublinear*.

By $\mathcal{A}(X_0, X_1)$ we denote the class of all admissible operators, i.e., subadditive operators $T : X_0 + X_1 \to X_0 + X_1$ such that the restrictions $T|_{X_i} : X_i \to X_i$ are bounded for $i = 1, 2$. Put

$$M_i := \|T|_{X_i}\|_{X_i \to X_i} = \sup_{x \in X_i, x \ne 0} \frac{\|Tx\|_{X_i}}{\|x\|_{X_i}}, \quad M := \max\{M_0, M_1\}.$$

A Banach space X is said to be an *interpolation space for subadditive operators* between Banach lattices X_0 and X_1 on the same measure space (Ω, Σ, μ) if

$$X_0 \cap X_1 \subset X \subset X_0 + X_1$$

with continuous embeddings, every operator $T \in \mathcal{A}(X_0, X_1)$ maps X into itself, and

$$\|T\|_{X \to X} \le CM.$$

The constant C is called an *interpolation constant*.

A typical example of a subadditive operator which is not linear is the Hardy-Littlewood maximal operator. It is well known that this operator is bounded on all Lebesgue spaces $L^p(\mathbb{R}^n)$, $1 < p \le \infty$. Its operator norm on $L^p(\mathbb{R})$, $1 < p < \infty$, was calculated only recently by Grafakos and Mongomery-Smith [5] (for dimensions $n \ge 2$, the problem is still open). A natural question about generalizations of those results to more general, for instance, Orlicz spaces, arises. This question was our particular motivation for the present work.

The aim of this paper is to study the interpolation of subadditive operators from the couple of Lebesgue spaces L^p and L^q to an Orlicz space with the special attention to the interpolation constant C. The corresponding problem for linear operators was considered in our paper [7]. The interpolation of sublinear and subadditive operators between general Banach lattices and for the Lions-Peetre real K-method and the Calderón complex method was considered in [13]. Despite a vast number of works on the interpolation of Orlicz spaces and their generalizations, only a few of them took care upon good estimates for the interpolation constant. We will not go into details here but refer to historical remarks in [7] and [14, Chapter 14].

A function $\rho : [0, \infty) \to [0, \infty)$ is said to be *quasi-concave* if it is continuous and positive on $\mathbb{R}_+ := (0, \infty)$ and

$$\rho(s) \le \max(1, s/t)\rho(t) \quad \text{for all} \quad s, t > 0.$$

Let $\mathcal{P}$ be the set of all quasi-concave functions and let $\widetilde{\mathcal{P}}$ denote the subset of all concave functions in $\mathcal{P}$. Note that if $\rho \in \mathcal{P}$, then its *concave majorant* $\widetilde{\rho}$ defined by

$$\widetilde{\rho}(t) := \inf_{s>0} \left(1 + \frac{t}{s}\right) \rho(s)$$

belongs to $\widetilde{\mathcal{P}}$ and

$$\rho(t) \le \widetilde{\rho}(t) \le 2\rho(t) \quad \text{for all} \quad t > 0. \tag{2}$$

The constant 2 is best possible.

Clearly, if $\theta \in (0,1)$, then $\rho(t) = t^\theta$ belongs to $\widetilde{\mathcal{P}}$. Let us give a nontrivial example of a function in $\widetilde{\mathcal{P}}$. For $0 < \theta < 1$ and $a, b \in \mathbb{R}$, let $\rho(0) = 0$ and $\rho(t) = t^\theta [\ln(e+t)]^a [\ln(e+1/t)]^b$ for $t > 0$. Then $\rho \in \widetilde{\mathcal{P}}$.

Following Gustavsson and Peetre [6] (see also [14, Chap. 14] and [7]), we suppose that

$$\varphi^{-1}(u) = u^{1/p} \rho(u^{1/q-1/p}) \quad \text{for all} \quad u > 0,$$

where $1 \le p < q \le \infty$ and $\rho \in \widetilde{\mathcal{P}}$. In this case φ is convex and the (well defined) Orlicz space L^φ is an intermediate space between L^p and L^q, that is,

$$L^p \cap L^q \subset L^\varphi \subset L^p + L^q$$

with both embeddings being continuous (see, e.g. [14, Lemma 14.2]).

The paper is organized as follows. Section 2 contains some information on the Peetre L-functional defined for a couple of Banach lattices. In Section 3, we study the limiting case of the interpolation between L^p and L^∞. The proof of the interpolation theorem is based on the Krée formula and the Hardy-Littlewood-Pólya majorization theorem. In Section 4, we embark on the generic case of the interpolation between L^p and L^q whenever $1 \le p < q < \infty$. Our approach goes back to Peetre [17]. The sharp estimate for the modified L-functional of the couple (L^p, L^q) due to Sparr [18] is the main ingredient of our proof. For completeness we also formulate a known interpolation theorem (see [7]) for linear operators in Section 5. It gives a slightly better estimate for the interpolation constant in the case $1 < p < q < \infty$ because in this case one can employ duality arguments.

2. Peetre L-functional on Banach lattices

The following Peetre L-functional plays a significant role in the real method of interpolation theory (see [17] and also [1, 18]). It is defined for $0 < p, q, t < \infty$ and for $x \in X_0 + X_1$ (where X_0 and X_1 are arbitrary Banach spaces, not necessarily having a lattice structure) by

$$K_{p,q}(t, x; X_0, X_1) := \inf \left\{ \|x_0\|_{X_0}^p + t\|x_1\|_{X_1}^q : \ x = x_0 + x_1, \ x_0 \in X_0, \ x_1 \in X_1 \right\}.$$

In the case when $p = q = 1$ this L-functional is the classical Peetre K-functional, which we shortly denote by $K(t, x; X_0, X_1)$.

Proposition 1. *Let $0 < p, q < \infty$ and let X_0, X_1 be (real or complex) Banach lattices on a complete σ-finite measure space (Ω, Σ, μ). If $t > 0$ and $x \in X_0 + X_1$, then $K_{p,q}(t, x; X_0, X_1)$ is equal to*

$$\inf \left\{ \|x_0\|_{X_0}^p + t\|x_1\|_{X_1}^q : \ |x| \le x_0 + x_1, \ 0 \le x_0 \in X_0, \ 0 \le x_1 \in X_1 \right\}.$$

Proof. If $x = x_0 + x_1$ with $x_0 \in X_0$ and $x_1 \in X_1$, then

$$|x| = x_0 e^{-i\theta} + x_1 e^{-i\theta} = y_0 + y_1,$$

where $\theta : \Omega \to \mathbb{R}$, and

$$K_{p,q}(t, |x|; X_0, X_1) \le \|y_0\|_{X_0}^p + t\|y_1\|_{X_1}^q = \|x_0\|_{X_0}^p + t\|x_1\|_{X_1}^q.$$

Hence

$$K_{p,q}(t, |x|; X_0, X_1) \le K_{p,q}(t, x; X_0, X_1). \tag{3}$$

Similarly, if $|x| = x_0 + x_1$ with $x_0 \in X_0$ and $x_1 \in X_1$, then

$$x = |x|e^{i\theta} = x_0 e^{i\theta} + x_1 e^{i\theta} = y_0 + y_1,$$

and

$$K_{p,q}(t, x; X_0, X_1) \le \|y_0\|_{X_0}^p + t\|y_1\|_{X_1}^q = \|x_0\|_{X_0}^p + t\|x_1\|_{X_1}^q,$$

from which we obtain the estimate

$$K_{p,q}(t, x; X_0, X_1) \le K_{p,q}(t, |x|; X_0, X_1). \tag{4}$$

Combining (3) and (4), we arrive at

$$K_{p,q}(t, x; X_0, X_1) = K_{p,q}(t, |x|; X_0, X_1). \tag{5}$$

Now let $|x| = x_0 + x_1 = \operatorname{Re} x_0 + \operatorname{Re} x_1$, where $x_0 \in X_0$ and $x_1 \in X_1$. Consider the sets

$$\begin{aligned} A_1 &:= \big\{t \in \Omega : \ \operatorname{Re} x_0(t) \ge 0, \ \operatorname{Re} x_1(t) \ge 0\big\}, \\ A_2 &:= \big\{t \in \Omega : \ \operatorname{Re} x_0(t) \ge 0, \ \operatorname{Re} x_1(t) < 0\big\}, \\ A_3 &:= \big\{t \in \Omega : \ \operatorname{Re} x_0(t) < 0, \ \operatorname{Re} x_1(t) \ge 0\big\}. \end{aligned}$$

Put

$$x_0'(t) := \begin{cases} \operatorname{Re} x_0(t) & \text{if} \quad t \in A_1, \\ \operatorname{Re} x_0(t) + \operatorname{Re} x_1(t) & \text{if} \quad t \in A_2, \\ 0 & \text{if} \quad t \in \Omega \setminus (A_1 \cup A_2), \end{cases}$$

$$x_1'(t) := \begin{cases} \operatorname{Re} x_1(t) & \text{if} \quad t \in A_1, \\ \operatorname{Re} x_0(t) + \operatorname{Re} x_1(t) & \text{if} \quad t \in A_3, \\ 0 & \text{if} \quad t \in \Omega \setminus (A_1 \cup A_3). \end{cases}$$

Since $\Omega \setminus (A_1 \cup A_2) = A_3$ and $\Omega \setminus (A_1 \cup A_3) = A_2$ (here we do not distinguish sets differing by a set of μ-measure zero) it follows that $|x| = x_0' + x_1'$ and $0 \le x_i' \le |\operatorname{Re} x_i| \le |x_i|$ for $i = 0, 1$. Thus the sets

$$\begin{aligned} S_1 &:= \big\{x \in X_0 + X_1 : \ |x| = x_0 + x_1, \ x_0 \in X_0, \ x_1 \in X_1\big\}, \\ S_2 &:= \big\{x \in X_0 + X_1 : \ |x| = x_0 + x_1, \ 0 \le x_0 \in X_0, \ 0 \le x_1 \in X_1\big\} \end{aligned}$$

coincide. If $x \in X_0 + X_1$ is such that $|x| \le x_0 + x_1$ with $0 \le x_0 \in X_0$ and $0 \le x_1 \in X_1$, then for $i = 0, 1$ put

$$y_i := \begin{cases} \frac{x_i |x|}{x_0 + x_1} & \text{if} \quad x_0 + x_1 > 0, \\ 0 & \text{if} \quad x_0 + x_1 = 0. \end{cases}$$

In that case $|x| = y_0 + y_1$ and $0 \le y_i \le x_i$. Hence the set

$$S_3 := \big\{x \in X_0 + X_1 : \ |x| \le x_0 + x_1, \ 0 \le x_0 \in X_0, \ 0 \le x_1 \in X_1\big\}$$

coincides with $S_2 = S_1$. Thus

$$K_{p,q}(t, |x|; X_0, X_1) = \inf_{x \in S_1} \left(\|x_0\|^p_{X_0} + t\|x_1\|^q_{X_1} \right) = \inf_{x \in S_3} \left(\|x_0\|^p_{X_0} + t\|x_1\|^q_{X_1} \right). \tag{6}$$

We finish the proof combining (5) and (6). □

The above statement allows us to study admissible subadditive operators on Banach lattices.

Proposition 2. *Let $0 < p, q < \infty$ and let X_0, X_1 be (real or complex) Banach lattices on a complete σ-finite measure space (Ω, Σ, μ). Suppose $T \in \mathcal{A}(X_0, X_1)$ and $x \in X_0 + X_1$. Then*

$$K_{p,q}\left(t, \frac{Tx}{M}; X_0, X_1\right) \leq K_{p,q}(t, x; X_0, X_1) \quad \textit{for all} \quad t > 0.$$

Proof. The proof is standard. If $x = x_0 + x_1$ is any decomposition of $x \in X_0 + X_1$ such that $x_0 \in X_0$ and $x_1 \in X_1$, then taking into account that T is subadditive, we have

$$\frac{|Tx|}{M} \leq \frac{|Tx_0|}{M} + \frac{|Tx_1|}{M}.$$

From Proposition 1 it follows that

$$\begin{aligned} K_{p,q}\left(t, \frac{Tx}{M}; X_0, X_1\right) &\leq \left\| \frac{|Tx_0|}{M} \right\|^p_{X_0} + \left\| \frac{|Tx_1|}{M} \right\|^q_{X_1} \\ &\leq \left(\frac{M_0}{M}\right)^p \|x_0\|^p_{X_0} + t\left(\frac{M_1}{M}\right)^q \|x_1\|^q_{X_1} \\ &\leq \|x_0\|^p_{X_0} + t\|x_1\|^q_{X_1}. \end{aligned}$$

Taking the infimum over all $x \in X_0 + X_1$ such that $x_0 \in X_0$ and $x_1 \in X_1$, we arrive at the desired inequality. □

3. Interpolation between L^p and L^∞

Our first main result is the following interpolation theorem.

Theorem 3. *Suppose $1 \leq p < \infty$.*

(a) *If $\psi(u) = \varphi(u^{1/p})$ is a convex function and $T \in \mathcal{A}(L^p, L^\infty)$, then*

$$I_\varphi\left(\frac{Tx}{2^{1-1/p}M}\right) \leq I_\varphi(x) \quad \textit{for all} \quad x \in L^p + L^\infty.$$

(b) *If $\varphi^{-1}(u) = u^{1/p}\rho(u^{-1/p})$ with $\rho \in \widetilde{\mathcal{P}}$ such that $\rho_*(\mathbb{R}_+) = \mathbb{R}_+$, where $\rho_*(t) := t\rho(1/t)$, then the Orlicz space L^φ (with both, the Luxemburg-Nakano and the Orlicz norm) is an interpolation space for subadditive operators between L^p and L^∞, and*

$$\|T\|_{L^\varphi \to L^\varphi} \leq C \max\left\{\|T\|_{L^p \to L^p}, \|T\|_{L^\infty \to L^\infty}\right\}$$

for any $T \in \mathcal{A}(L^p, L^\infty)$, where $C \leq 2^{1-1/p}$.

Proof. (a) The proof is developed by analogy with the proof of [7, Theorem 4.2(b)]. For all $x \in L^p + L^\infty$ and $t > 0$, according to the Krée formula (see [4, Theorem 5.2.1]), we have

$$\left(\int_0^t x^*(s)^p ds\right)^{1/p} \leq K(t^{1/p}, x; L^p, L^\infty) \leq 2^{1-1/p}\left(\int_0^t x^*(s)^p ds\right)^{1/p}. \tag{7}$$

Notice that the constant $2^{1-1/p}$ on the right-hand side is best possible (see Bergh [3]). Due to Proposition 2,

$$K\left(t, \frac{Tx}{M}; L^p, L^\infty\right) \leq K(t, x; L^p, L^\infty) \quad \text{for all} \quad t > 0. \tag{8}$$

From (7) and (8) we obtain that

$$\int_0^t \left(\frac{(Tx)^*(s)}{2^{1-1/p}M}\right)^p ds \leq \int_0^t x^*(s)^p\, ds \quad \text{for all} \quad t > 0.$$

Since $\psi(u) = \varphi(u^{1/p})$ is convex it follows, by the Hardy-Littlewood-Pólya majorization theorem (see, e.g. [2, p. 88]) and equality (1), that

$$\begin{aligned}
\int_\Omega \varphi\left(\frac{|Tx(t)|}{2^{1-1/p}M}\right) d\mu(t) &= \int_0^\infty \varphi\left(\frac{(Tx)^*(s)}{2^{1-1/p}M}\right) ds \\
&= \int_0^\infty \psi\left(\left[\frac{(Tx)^*(s)}{2^{1-1/p}M}\right]^p\right) ds \\
&\leq \int_0^\infty \psi(x^*(s)^p)\, ds \\
&= \int_0^\infty \varphi(x^*(s))\, ds \\
&= \int_\Omega \varphi\left(|x(t)|\right) d\mu(t),
\end{aligned}$$

and this is a desired statement. Part (a) is proved.

(b) It is possible to prove that if $\varphi^{-1}(u) = u^{1/p}\rho(u^{-1/p})$ then the function $\psi(u) = \varphi(u^{1/p})$ is convex (cf. [7, Lemma 3.2(d)] for details). Hence, by part (a), we obtain the modular estimate

$$I_\varphi\left(\frac{Tx}{2^{1-1/p}M}\right) \leq I_\varphi(x) \quad \text{for all} \quad x \in L^\varphi,$$

which implies

$$\|Tx\|_\varphi \leq 2^{1-1/p}M\|x\|_\varphi \quad \text{and} \quad \|Tx\|_\varphi^0 \leq 2^{1-1/p}M\|x\|_\varphi^0$$

for all $x \in L^\varphi$. $\square$

Theorem 3 was proved for linear operators in our paper [7]. In the case $p = 1$, Theorem 3(b) generalizes the well-known Orlicz interpolation theorem to subadditive operators. Orlicz proved it in 1934 for linear operators and with certain constant $C > 1$. From the Calderón-Mitjagin interpolation theorem (see, e.g. [9, Chap. 2, Theorem 4.9]) it follows that, in fact, the interpolation constant is equal to 1. A simple proof of the Orlicz interpolation theorem for linear and for Lipschitz operators with the interpolation constant 1, together with its applications, was given by one of the authors [12] (see also [14]). Lorentz and Shimogaki [10, Theorem 7] observed that if $1 \leq p < \infty$, $\varphi(u) = \int_0^u (u-t)^p\, dm(t)$, where

$m : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function, and $T \in \mathcal{A}(L^p, L^\infty)$ is a linear operator, then $\|T\|_{L^\varphi \to L^\varphi} \le \max\{\|T\|_{L^p}, \|T\|_{L^\infty}\}$.

4. Interpolation between the Lebesgue spaces L^p and L^q with $1 \le p < q < \infty$

We will need the following properties of convex and concave functions.

Lemma 4. *Suppose that $1 \le p < q < \infty$ and, for some $\rho \in \widetilde{\mathcal{P}}$,*

$$\varphi^{-1}(u) = u^{1/p}\rho(u^{1/q-1/p}) \quad \text{for all} \quad u > 0.$$

Then φ is convex and there exists a function $h \in \mathcal{P}$ such that

$$\varphi(u) = u^q h(u^{p-q}) \quad \text{for all} \quad u > 0.$$

Proof. For a proof, see [7, Lemma 3.2(b)]. □

Note that the above lemma guarantees only that $h \in \mathcal{P}$ and h need not necessarily to be concave. We illustrate this observation with the following simple example.

Example 5. If $1 \le p < q < \infty$ and $\varphi^{-1}(u) = u^{1/p}\rho(u^{1/q-1/p})$ with $\rho(t) = \min\{1, t\}$, then $\varphi(u) = u^q h(u^{p-q})$ with $h(t) = \max\{1, t\}$. Obviously, $\rho \in \widetilde{\mathcal{P}}$ and $h \in \mathcal{P} \setminus \widetilde{\mathcal{P}}$.

Lemma 6 (Peetre, 1966)**.** *Every function $h \in \widetilde{\mathcal{P}}$ can be represented in the form*

$$h(u) = a_h + b_h u + \int_0^\infty \min\{u, t\}\, dm(t) \quad \text{for all} \quad u > 0, \tag{9}$$

where

$$a_h := \lim_{u \to 0+} h(u), \quad b_h := \lim_{u \to \infty} \frac{h(u)}{u}, \tag{10}$$

and $m : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function (in fact, $m(t) = -h'(t)$).

Proof. A proof of this result is contained in [16], [4, Lemma 5.4.3]. □

We consider the modified Peetre L-functional $K^*_{p,q}$ for the couple of Lebesgue spaces (L^p, L^q) defined by

$$K^*_{p,q}(t, x; L^p, L^q) := \int_\Omega \min\left\{|x(s)|^p, t|x(s)|^q\right\} d\mu(s).$$

Lemma 7 (Sparr, 1978)**.** *Suppose $1 \le p < q < \infty$. If $x, y \in L^p + L^q$ and*

$$K_{p,q}(t, x; L^p, L^q) \le K_{p,q}(t, y; L^p, L^q) \quad \text{for all} \quad t > 0,$$

then

$$K^*_{p,q}(t, x; L^p, L^q) \le \gamma_{p,q} K^*_{p,q}(t, y; L^p, L^q) \quad \text{for all} \quad t > 0,$$

where

$$\gamma_{p,q} := \inf\left\{\gamma > 0 \;:\; \inf_{\substack{x + y = \gamma, \\ x, y \ge 0}} (x^p + y^q) = 1\right\}.$$

The constant $\gamma_{p,q}$ cannot be replaced by any smaller constant.

Proof. For a proof, see [18, Lemma 5.1 and Example 5.3]. □

We have found natural to attribute Sparr's name to the constants $\gamma_{p,q}$. Sparr observed that $1 < \gamma_{p,q} < 2$. The Sparr constants play an important role in our final result. Now we give some more precise information about the Sparr constants.

Proposition 8 (Karlovich-Maligranda, 2001)**.** *Let $1 \le p, q < \infty$.*

(a) *We have $\gamma_{p,q} = \gamma_{q,p}$ and $\gamma_{1,1} = 1$.*
(b) *If $q > 1$, then*

$$\gamma_{p,q} = \inf\left\{ x + \left(\frac{p}{q}x^{p-1}\right)^{1/(q-1)} \ : \ x^p + \left(\frac{p}{q}x^{p-1}\right)^{q/(q-1)} = 1 \right\}.$$

In particular, $\gamma_{q,q} = 2^{1-1/q}$ and $\gamma_{1,q} = 1 + q^{1/(1-q)} - q^{q/(1-q)}$.
(c) *$\gamma_{p,q}$ continuously increases in p and q.*
(d) *If $p \le q$, then $2^{1-1/p} \le \gamma_{p,q} \le 2^{1-1/q}$.*

Proof. A proof can be found in [7, Proposition 4.3]. □

We are ready to prove our second main result: the modular estimate and the estimate for the norm of an admissible operator $T \in \mathcal{A}(L^p, L^q)$.

Theorem 9. *Suppose $1 \le p < q < \infty$.*

(a) *If $\varphi(u) = u^q h(u^{p-q})$ for some $h \in \widetilde{\mathcal{P}}$ and $T \in \mathcal{A}(L^p, L^q)$, then*

$$I_\varphi\left(\frac{Tx}{M}\right) \le \gamma_{p,q} I_\varphi(x) \quad \text{for all} \quad x \in L^p \cap L^q.$$

(b) *If $\varphi^{-1}(u) = u^{1/p}\rho(u^{1/q-1/p})$ for some $\rho \in \widetilde{\mathcal{P}}$. Then the Orlicz space L^φ (with both, the Luxemburg and the Orlicz norm) is an interpolation space for subadditive operators between L^p and L^q, and*

$$\|T\|_{L^\varphi \to L^\varphi} \le C \max\left\{\|T\|_{L^p \to L^p}, \|T\|_{L^q \to L^q}\right\}$$

for any $T \in \mathcal{A}(L^p, L^q)$, where $C \le (2\gamma_{p,q})^{1/p} \le 2^{(2-1/q)/p} < 4$.

Proof. (a) The idea of the proof goes back to Peetre [17]. We follow the proof of [7, Theorem 4.2(a)]. Due to Lemma 6, the function h can be represented in the form (9). Hence,

$$\varphi(u) = u^q h(u^{p-q}) = a_h u^q + b_h u^p + \int_0^\infty \min\{u^p, tu^q\}\, dm(t) \tag{11}$$

for all $u \in \mathbb{R}_+$. Consequently,

$$\begin{aligned} I_\varphi\left(\frac{Tx}{M}\right) &= \int_\Omega \varphi\left(\frac{|Tx(s)|}{M}\right) d\mu(s) \\ &= a_h \left\|\frac{Tx}{M}\right\|_q^q + b_h \left\|\frac{Tx}{M}\right\|_p^p \\ &\quad + \int_\Omega \left[\int_0^\infty \min\left\{\left(\frac{|Tx(s)|}{M}\right)^p, t\left(\frac{|Tx(s)|}{M}\right)^q\right\} dm(t)\right] d\mu(s). \end{aligned}$$

Since the operator T is bounded in L^p and L^q, we get

$$\begin{aligned} a_h \left\|\frac{Tx}{M}\right\|_q^q + b_h \left\|\frac{Tx}{M}\right\|_p^p &\le a_h \left(\frac{M_1}{M}\right)^q \|x\|_q^q + b_h \left(\frac{M_0}{M}\right)^p \|x\|_p^p \\ &\le a_h \|x\|_q^q + b_h \|x\|_p^p \end{aligned}$$

and, according to Proposition 2,

$$K_{p,q}\left(t, \frac{Tx}{M}; L^p, L^q\right) \le K_{p,q}(t, x; L^p, L^q) \quad \text{for all} \quad t > 0.$$

Applying Sparr's Lemma 7 we obtain

$$K^*_{p,q}\left(t, \frac{Tx}{M}; L^p, L^q\right) \le \gamma_{p,q} K^*_{p,q}(t, x; L^p, L^q) \quad \text{for all} \quad t > 0.$$

Hence, by the Fubini theorem and in view of the definition of $K^*_{p,q}$, we conclude that

$$\begin{aligned}
&\int_\Omega \left[\int_0^\infty \min\left\{\left(\frac{|Tx(s)|}{M}\right)^p, t\left(\frac{|Tx(s)|}{M}\right)^q\right\} dm(t)\right] d\mu(s) \\
&= \int_0^\infty K^*_{p,q}\left(t, \frac{Tx}{M}; L^p, L^q\right) dm(t) \\
&\le \gamma_{p,q} \int_0^\infty K^*_{p,q}(t, x; L^p, L^q)\, dm(t) \\
&= \gamma_{p,q} \int_\Omega \left[\int_0^\infty \min\left\{|x(s)|^p, t|x(s)|^q\right\} dm(t)\right] d\mu(s).
\end{aligned}$$

Combining the above estimates and taking into account that $\gamma_{p,q} > 1$ we obtain

$$\begin{aligned}
I_\varphi\left(\frac{Tx}{M}\right) &\le a_h\|x\|_q^q + b_h\|x\|_p^p \\
&\quad + \gamma_{p,q} \int_\Omega \left[\int_0^\infty \min\left\{|x(s)|^p, t|x(s)|^q\right\} dm(t)\right] d\mu(s) \\
&\le \gamma_{p,q} \Bigg(a_h\|x\|_q^q + b_h\|x\|_p^p \\
&\quad + \int_\Omega \left[\int_0^\infty \min\left\{|x(s)|^p, t|x(s)|^q\right\} dm(t)\right] d\mu(s)\Bigg) \\
&= \gamma_{p,q} \int_\Omega \varphi(|x(s)|)\, d\mu(s) \\
&= \gamma_{p,q} I_\varphi(x).
\end{aligned}$$

Part (a) is proved.

(b) This statement is proved by analogy with [7, Theorem 5.1 (a)-(b)]. From [14, Lemma 14.2] it follows that the function φ is convex. Hence the Orlicz space L^φ is well defined.

By Lemma 4, there is a function $h \in \mathcal{P}$ such that $\varphi(u) = u^q h(u^{p-q})$. From (2) we see that $\widetilde{h} \in \widetilde{\mathcal{P}}$ and

$$\varphi(u) \le u^q \widetilde{h}(u^{p-q}) \le 2\varphi(u) \quad \text{for all} \quad u > 0. \tag{12}$$

Applying Theorem 9(a) to the function $\psi(u) = u^q \widetilde{h}(u^{p-q})$ and taking into account (12), we obtain

$$I_\varphi\left(\frac{Tx}{M}\right) \le I_\psi\left(\frac{Tx}{M}\right) \le \gamma_{p,q} I_\psi(x) \le 2\gamma_{p,q} I_\varphi(x) \tag{13}$$

for all $x \in L^p \cap L^q$. From the properties of h one can conclude that φ satisfies the Δ_2-condition on $[0,\infty)$ and

$$I_\varphi\left(\frac{Tx}{(2\gamma_{p,q})^{1/p}M}\right) \le \frac{1}{2\gamma_{p,q}} I_\varphi\left(\frac{Tx}{M}\right) \le I_\varphi(x)$$

for all $x \in L^p \cap L^q$. Hence

$$\|Tx\|_\varphi \le (2\gamma_{p,q})^{1/p} M \|x\|_\varphi, \quad \|Tx\|^0_\varphi \le (2\gamma_{p,q})^{1/p} M \|x\|^0_\varphi \tag{14}$$

for all $x \in L^p \cap L^q$. Since φ satisfies the Δ_2-condition for all $u \ge 0$, it follows that $L^p \cap L^q$ is dense in L^φ (for the case of Orlicz spaces generated by N-functions and defined on Euclidean spaces of finite measure, see [8, Chap. 2]; the proof in a more general situation considered in this paper is analogous). Thus, inequalities (14) are fulfilled for all $x \in L^\varphi$. This fact and Proposition 8(d) show that $C \le (2\gamma_{p,q})^{1/p} \le 2^{(2-1/q)/p} < 4$. $\square$

Remark 10. If $1 \le p < q < \infty$ and $\varphi(u) = u^q h(u^{p-q})$ with $h \in \widetilde{\mathcal{P}}$, then from the proof of the above theorem it follows that L^φ is an interpolation space between L^p and L^q, and we have a better estimate of the interpolation constant:

$$C \le (\gamma_{p,q})^{1/p} \le 2^{1/(q'p)} < 2.$$

As it was pointed out by Mastyło [15], Example 5.4 in our paper [7] illustrating this possibility is erroneous. We substitute it by the following.

Example 11. The function h given by $h(0) = 0$ and $h(t) = t\ln(1+1/t)$ for $t > 0$ belongs to $\widetilde{\mathcal{P}}$ and for all p, q satisfying $1 \le p < q < \infty$ the function

$$\varphi(u) = u^q h(u^{p-q}) = u^p \ln(1+u^{q-p})$$

is convex on $[0,\infty)$.

Indeed, for $t > 0$ we have

$$h'(t) = \ln\left(1+\frac{1}{t}\right) - \frac{1}{1+t} > 0, \quad h''(t) = \frac{-1}{t(1+t)^2} < 0,$$

thus h is increasing and concave on $\mathbb{R}_+$. On the other hand, simple calculations give

$$\begin{aligned} \varphi''(u) &= p(p-1)u^{p-2}\ln(1+u^{q-p}) + p(q-p)\frac{u^{q-2}}{1+u^{q-p}} \\ &\quad + (q-p)\frac{(q-1)u^{q-2} + (p-1)u^{2q-p-2}}{(1+u^{q-p})^2}. \end{aligned}$$

Since $p \ge 1$ and $q > p$, we have $\varphi''(u) \ge 0$ for all $u \ge 0$. Thus φ is convex on $[0,\infty)$.

5. Interpolation of linear operators

The interpolation constant in Theorem 9(b) can be improved for linear operators by using the duality argument. By p' we denote the conjugate number to p, $1 < p < \infty$, defined by $1/p + 1/p' = 1$.

Theorem 12 (Karlovich-Maligranda, 2001)**.** *Let $1 < p < q < \infty$ and*

$$\varphi^{-1}(u) = u^{1/p}\rho(u^{1/q-1/p})$$

for some $\rho \in \widetilde{\mathcal{P}}$*. Then the Orlicz space* L^φ *(with both, the Luxemburg-Nakano and the Orlicz norm) is an interpolation space for linear operators between* L^p *and* L^q*, and*

$$\|T\|_{L^\varphi \to L^\varphi} \le C \max\left\{\|T\|_{L^p \to L^p}, \|T\|_{L^q \to L^q}\right\}$$

for any admissible linear operator $T \in \mathcal{A}(L^p, L^q)$*, where*

$$C \le \min\left\{(2\gamma_{p,q})^{1/p}, (2\gamma_{q',p'})^{1/q'}\right\} \le 2^{1/(pq')+\min\{1/p,1/q'\}} < 4.$$

In particular, if either $1 < p < q \le 2$ *or* $2 \le p < q < \infty$*, then* $C < 2$*.*

Proof. This result is proved in [7, Theorem 5.1(b)]. □

The estimates we proved above can be used in the norm estimation of some concrete linear operators (like Hardy operators, convolution operators, integral operators, the Hilbert transform or other singular integral operators) and subadditive operators (like maximal operators) between Orlicz spaces.

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